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# On the application of the Helmholtz–Hodge decomposition in projection methods for incompressible flows with general boundary conditions

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SUMMARY

This paper is concerned with the analysis of the Helmholtz–Hodge decomposition theorem since it plays a fundamental role in the projection methods that are adopted in the numerical solution of the Navier–Stokes equations for incompressible flows. The paper highlights the role of the orthogonal decomposition of a vector field in a bounded domain when general boundary conditions are in effect. In fact, even if Fractional Time-Step Methods are standard procedures for de-coupling the pressure gradient and the velocity field, many problems are encountered in performing the decoupling with higher accuracy. Since the problem of determining a unique and orthogonal decomposition requires only one boundary condition to be well posed, thus either the normal or the tangential ones, result exactly imposed at the end of the projection. Numerical errors are introduced in terms of both the pressure and the velocity but the orthogonality of decomposition guarantees that the former does not contribute to affect the accuracy of the latter. Moreover, it is shown that depending on the meaning of the vector to be decomposed, i.e. acceleration or velocity, the true orthogonal projector can be defined only when suitable boundary conditions are verified. Conversely, it is shown that when the decomposition results non-orthogonal, the velocity accuracy suffers of other errors. The issue on the resulting accuracy order of the procedure is clearly addressed by means of several accuracy studies and a strategy for improving it is proposed. This paper follows and integrates the issues reported in Iannelli and Denaro (*Int. J. Numer. Meth. Fluids* 2003; **42**:399–437). Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: Helmholtz–Hodge decomposition; incompressible flows; fractional time-step methods; boundary conditions for projection methods

## 1. INTRODUCTION

The Helmholtz decomposition theorem states that *a smooth vector field* **w** *is uniquely determined*, *in a bounded regular domain*, *when its divergence*, *curl and normal* (*or tangential*) *component on the boundary*, *are assigned*. Consequently, it is implied from this theorem that every smooth vector field decomposes as the sum of a gradient vector field and one, which

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is divergence-free. On a compact Riemannian manifold, this splitting becomes the classical Hodge decomposition theorem. In particular, the theorem guarantees that given a compact Riemannian manifold $R$ and a differential form $\varpi$ then there exist three differential forms, one of them being a harmonic differential one, for which $\varpi$ expresses as sum of them. The difference in Helmholtz's decomposition expressed by two terms and the one defined with three terms by Hodge results from the compactness of the manifold $R$. The Hodge decomposition theorem implies that every smooth differential form on a compact manifold decomposes into the sum of three parts, each one being again smooth. However, Hodge decomposition holds in much greater generality.

The Helmholtz–Hodge decomposition theorem (also known as Ladyzhenskaja theorem and indicated as HHD in the following) plays a basic role in the theory of generalized solutions[‡] as well as in the numerical approximation of physical models as the Navier–Stokes equations [1–9]. One of the major difficulties that are encountered in solving the system of differential equations for isothermal incompressible flows resides in the fact that the velocity vector and the pressure gradient fields result coupled each other by the continuity constraint, because the pressure is only a Lagrange multiplier, not a thermodynamic state variable. As, for solving continuity and momentum equations (resulting in a Stokes-like system), this coupling would lead to use heavy computational procedures, splitting methodologies are often implemented. Owing to their simplicity in de-coupling the problem and solving separately the parabolic/elliptic equations, fractional methods are massively used for several cases [10–18].

The so-called Fractional Time-Step Method (FTSM) provides the solution of the incompressible Navier–Stokes equations in certain separated steps. The first one is simply based on the solution of the time-discretized momentum equation with (referred as *incremental-pressure projection methods*) or without (referred as *pressure-free projection methods*) a provisional pressure gradient. Such equation is associated to a suitable set of numerical boundary conditions for parabolic-type equations and provides a non-solenoidal intermediate velocity vector, say $\mathbf{v}^*$, which is afterwards projected onto the space of divergence-free vector functions. Hence, the second step consists in solving the Poisson equation associated to proper closure conditions while, in order to enforce the continuity, the last step is the correction of $\mathbf{v}^*$ by means of the computed gradient field. Therefore, considering the HHD as being a part of the FTSM, one can summarize that the first step provides the vector function $\mathbf{w} = \mathbf{v}^*$ then projected *onto the space of divergence-free vector functions having either vanishing normal component along the boundary* (i.e. parallel to the frontier of the domain) *or having the tangential components assigned on the boundary*. According to the HHD enunciate, the projector is an orthogonal operator, i.e. is symmetric with respect to the inner product of two vectors, bounded and idempotent, gradient and divergence operators result adjoints and the decomposition is unique. Although several second-order time accurate projection methods were proposed and the guidelines of the FTSM are well understood, it appears still largely debated in the literature [19–28] how the de-coupling affects the actual accuracy of the original Chorin's method who showed only a first-order convergence rate of the solution [29].

[‡]Hermann Weyl revealed an error in the Hodge's proof of the existence theorem for harmonic forms, originally published in the book in 1941. Although the 1952 edition of Hodge's text corrects this, Kodaira is also credited to have provided the first correct argument.

Apart from the global accuracy of projection methods, which is analysed in many papers, a controversial issue consists in the proper prescription of the boundary conditions. In fact, for example considering the case of homogeneous Dirichlet conditions, the orthogonal projection of $\mathbf{v}^*$, associated to vanishing normal component, ensures that the resulting divergence-free vector accomplishes the normal condition on the boundary, while the fulfilment of the null tangential ones is not ensured. Similarly, if the null tangential components are assigned in the orthogonal decomposition the same remark remains valid, since the tangential will be satisfied whereas the normal component is not ensured to vanish. In other words, the HHD does not guarantee that the obtained divergence-free vector simultaneously accomplishes all the physical values on the boundaries. This happens because the HHD problem requires only one boundary condition to be well posed (either the normal velocity component or the vector tangential one), the others resulting in a way, which depends onto the approximations in computing $\mathbf{v}^*$. Generally, in order to remedy this approximation, for example for imposed normal component, the obtained tangential components are simply disregarded at the end of each time step and reset to their known values on the boundary. However, this strategy of resetting the tangential component to its correct physical value was proved to remain still inadequate, as well as it can reduce the smoothness of the velocity field [14, 23].

A further remark is that the FTSM has been largely applied in flows condition having non-vanishing as well as non-steady normal components, i.e. with non-homogeneous Dirichlet conditions (e.g. simulations of backward facing-step flows). As there are not vanishing normal components on the frontier, the general assertion of the HHD theorem would fail. Nevertheless, these approaches are considered, in a more general sense, as extensions of the HHD method [14, 20], even if they do not retain the same properties of the proper orthogonal decomposition. For example, in the recent paper of Brown *et al.* [14], the Hodge decomposition of the momentum equation with non-homogeneous velocity boundary conditions (see Equation (29) of that paper), is reported. Thus, the intermediate vector $\mathbf{v}^*$ is projected onto a subspace of divergence-free vector functions but, this time, having a *non-vanishing normal component* on the boundary. This fact implies that the computed divergence-free velocity field is no longer the unique vector component of the HHD but is only a component of a more general decomposition along non-orthogonal (apart from some specific cases) subspaces given by the momentum equation terms. As reported in Reference [20] '*the question to be addressed is whether orthogonality is really important*'; in other words, in order to adopt general boundary conditions, one asks if one can resort to a non-orthogonal decomposition or the proper orthogonal projection is a necessary condition. In such cases, it is important to analyse the consequences that appear because many test cases or analytical solutions with such general boundaries are adopted for testing new splitting algorithms and the results can be misleading. A consideration on this question will be addressed in the following.

The paper is organized as follows. First, in Section 2 the mathematical outlines of the HHD theorem are reported. This section is addressed, for the sake of completeness, to a reader not necessarily familiar with this theorem, but can be someway overstepped by other readers. Sections 3 and 4 are the core of the paper wherein the implications of the HHD theorem in solving the Navier–Stokes system with the FTSM, are analysed. Specifically, Section 3 focuses on the determination of the Eulerian acceleration $\mathbf{a}$ by means of the HHD of an acceleration $\mathbf{a}^*$ at an initial time $t_0$ whereas Section 4 illustrates the determination in the *pressure-free* *projection* method [14] of the orthogonal decomposition of $\mathbf{v}^*$ which is solution of a time discretized equation, traditionally obtained by applying the Adams-Bashforth/Crank-Nicolson (AB/CN) scheme in the prediction step.

In fact, it is shown in Section 3 that a non-solenoidal acceleration $\mathbf{a}^*$ can be preliminary computed, from the differential momentum equation without time discretization. Since at the initial time the velocity vector is assumed to be given everywhere over the bounded domain, the vector $\mathbf{a}^*(t_0)$ takes into account all the correct boundary conditions. In order to determine both the acceleration vector $\mathbf{a}(t_0)$, with prescribed vanishing normal component on the boundary, and the pressure gradient, the HHD can be directly applied to the momentum equation, before the time integration. Then, an initial value problem can be solved for obtaining the divergence-free velocity field at the new time. Several flow cases are illustrated to clarify the procedure and extend the HHD criterion to the case of non-vanishing normal component.

Conversely, in Section 4 it is highlighted how the field $\mathbf{v}^*$ depends on the implicit time integration of the initial velocity $\mathbf{v}(t_0)$ therefore, the computation already suffers by pressure errors. An approximation of the boundary conditions for the auxiliary variables is required and the successive HHD cannot remedy to the tangential assignment.

Hence, the meaning of $\mathbf{a}^*$ deeply differs from that of the vector $\mathbf{v}^*$ ($\mathbf{a}^*$ is not the time derivative of $\mathbf{v}^*$): both are mathematical positions indicating an intermediate field, but the first one posed as sum of the initial diffusive and advective differential terms of the momentum equation, the latter as a provisional updated velocity. This way, it is shown that the HHD of $\mathbf{a}^*$, differently from that of $\mathbf{v}^*$, extends to channel flows and steady boundary conditions. It is clarified that, for other kind of flows, the decomposition is not necessarily orthogonal, the uniqueness being no longer ensured (apart from particular cases, for example with the pressure constant on the boundary). The theoretical conclusions are further validated by simple numerical tests illustrated in a section. The results put in evidence the relevance of orthogonality into the decomposition and address how to perform the projection with second-order time-accuracy with general boundary conditions.

## 2. HELMHOLTZ–HODGE ORTHOGONAL DECOMPOSITION THEOREM

In this section, some outlines of the Helmholtz–Hodge theorem on the orthogonal decomposition for continuous operators are provided. A reader already familiar with this issue can someway overstep to the next sections.

The work of Hodge leads to a generalized Laplace operator for functions and differential forms defined on Riemannian manifolds, and Kodaira aptly described Hodge's theory of harmonic integrals as generalized potential theory. The most essential result in Hodge theory guarantees the existence and uniqueness of a solution to Laplace's equation subject to constraints given as integral equations on a compact Riemannian manifold. Thus, Hodge theory gives a fundamental connection between partial differential operators and topology.[§]

[§]These integral equations may also be seen as determining a *de Rham cohomology class* and the existence and uniqueness results imply that each cohomology class may be represented by a unique harmonic differential form. Another of such connection comes from Stokes' theorem which uses the exterior derivative to relate an integral over a domain to an integral over its boundary.

The Hodge decomposition theorem gives a splitting of any differential form into the sum of three components, the properties of which are strongly tied to Laplace's equation. The Helmholtz theorem states that every smooth vector field decomposes as the sum of a gradient vector field and one, which is divergence-free.

The statement is that:

*Theorem*
A given vector field $\mathbf{w}$ is uniquely decomposed, in a bounded domain $\Omega$ with smooth boundary $\partial\Omega$, in a pure gradient field and a divergence-free vector parallel to $\partial\Omega$.

It also follows that the vector field $\mathbf{w}$ ($\mathbf{w}\in\mathbf{L}^2(\Omega)$ having denoted by $\mathbf{L}^2(\Omega)$ the space of vector functions whose square modulus is integrable in $\Omega$ and by $H^1(\Omega)$ the space of $\mathbf{L}^2(\Omega)$ functions with first derivative in $\mathbf{L}^2(\Omega)$) *is uniquely determined when its divergence* (say $\rho$) *and its* curl (*say* $\mathbf{r}$) *are assigned along with the normal* (*or tangential*) *component on the boundary*.

Thus, the following equations must be satisfied:

$$
\begin{aligned}
&\underline{\nabla}\cdot\mathbf{w}=\rho\\
&\underline{\nabla}\wedge\mathbf{w}=\mathbf{r},\quad \mathbf{x}\in\Omega
\end{aligned}
\tag{1}
$$

with either the boundary condition

$$
\begin{aligned}
&\mathbf{n}\cdot\mathbf{w}=w_n\\
&\text{or:}\\
&\mathbf{n}\wedge\mathbf{w}=\mathbf{w}_t,\quad \mathbf{x}\in\partial\Omega
\end{aligned}
$$

$\mathbf{n}$ being the unit vector outward to $\partial\Omega$, $w_n$ and $\mathbf{w}_t$ the prescribed normal (scalar) component and tangential vector component of $\mathbf{w}$ on $\partial\Omega$, respectively.

As above addressed, $\mathbf{w}$ can be represented as the sum of $\mathbf{w}_1=\underline{\nabla}\varphi$, $\varphi\in H^1(\Omega)$ and $\mathbf{w}_2=\underline{\nabla}\wedge\mathbf{b}$, $\mathbf{w}_2\in\mathbf{L}^2(\Omega)$, being $\mathbf{b}$ a solenoidal vector; thus, according to (1) one has that the two components of the HHD must satisfy:

$$
\mathbf{w}=\mathbf{w}_1+\mathbf{w}_2=\underline{\nabla}\varphi+\underline{\nabla}\wedge\mathbf{b}
\tag{2}
$$

$$
\begin{gathered}
\underline{\nabla}\cdot\mathbf{w}_1=\rho\\
\underline{\nabla}\wedge\mathbf{w}_2=\mathbf{r},\quad \mathbf{x}\in\Omega\\
\mathbf{n}\cdot\mathbf{w}_2=0\Rightarrow\mathbf{w}_2=\mathbf{w}_{2t}\Rightarrow\mathbf{n}\cdot\underline{\nabla}\varphi=w_n\\
\text{or:}\\
\mathbf{n}\wedge\mathbf{w}_1=\mathbf{0}\Rightarrow\mathbf{w}_{1t}=\mathbf{0}\Rightarrow\mathbf{n}\wedge(\underline{\nabla}\wedge\mathbf{b}):=\mathbf{w}_{2t}=\mathbf{w}_t,\quad \mathbf{x}\in\partial\Omega
\end{gathered}
$$

Now, let us briefly proof orthogonality, existence and uniqueness of the decomposition $(2)_1$ for both cases of the normal $(2)_4$ or the tangential $(2)_5$ assignments.

The *orthogonality* of the vectors, in the sense of the inner product, is verified from

$$\begin{aligned}\int_\Omega \mathbf{w}_1\cdot\mathbf{w}_2\,\mathrm{d}V &= \int_\Omega \underline{\nabla}\varphi\cdot\mathbf{w}_2\,\mathrm{d}V\\ &= \int_\Omega \underline{\nabla}\cdot(\varphi\mathbf{w}_2)\,\mathrm{d}V - \int_\Omega \varphi\underline{\nabla}\cdot\mathbf{w}_2\,\mathrm{d}V\\ &= \int_{\partial\Omega} \varphi\mathbf{n}\cdot\mathbf{w}_2\,\mathrm{d}S = 0 \quad \text{from } (2)_4\end{aligned}$$

$$\begin{aligned}\int_\Omega \mathbf{w}_1\cdot\mathbf{w}_2\,\mathrm{d}V &= \int_\Omega \underline{\nabla}\varphi\cdot(\underline{\nabla}\wedge\mathbf{b})\,\mathrm{d}V\\ &= \int_\Omega \underline{\nabla}\cdot(\mathbf{b}\wedge\underline{\nabla}\varphi)\,\mathrm{d}V\\ &= \int_{\partial\Omega} \mathbf{n}\cdot(\mathbf{b}\wedge\underline{\nabla}\varphi)\,\mathrm{d}S\\ &= \int_{\partial\Omega} \mathbf{b}\cdot(\underline{\nabla}\varphi\wedge\mathbf{n})\,\mathrm{d}S = 0 \quad \text{from } (2)_5\end{aligned} \qquad (3)$$

in case one prescribes normal or tangential condition, respectively.

Therefore, for both cases, there exists an orthogonal[¶] projection operator denoted as follows:

⇒ $P_H$ onto $H$ such that $P_H(\mathbf{w})=\mathbf{w}_2$, where $\mathbf{w}_2$ defined in the divergence-free space $H=\{\mathbf{v}\in\mathbf{L}^2(\Omega)\colon\ \underline{\nabla}\cdot\mathbf{v}=0,\ \mathbf{n}\cdot\mathbf{v}=0 \text{ on } \partial\Omega\}$ if the normal component $(1)_3$ is assigned, or

⇒ $P_{H'}$ onto $H'$ such that $P_{H'}(\mathbf{w})=\mathbf{w}_2$, being $\mathbf{w}_2$ defined in the divergence-free space $H'=\{\mathbf{v}\in\mathbf{L}^2(\Omega)\colon\ \underline{\nabla}\cdot\mathbf{v}=0, (\mathbf{n}\wedge\mathbf{v}-\mathbf{w}_t)=\mathbf{0} \text{ on } \partial\Omega\}$, the tangential vector component $(1)_4$ being assigned, respectively.

Actually, observe that to have orthogonality ensured, the condition $\mathbf{n}\cdot\mathbf{w}_2=0$, as well as the condition $\mathbf{n}\wedge\mathbf{w}_1=\mathbf{0}$, is sufficient but not necessary, being for the two cases only required either $\int_{\partial\Omega}\varphi\mathbf{n}\cdot\mathbf{w}_2\,\mathrm{d}S=0$ or $\int_{\partial\Omega}\mathbf{b}\cdot(\underline{\nabla}\varphi\wedge\mathbf{n})\,\mathrm{d}S=0$, respectively. For example, if $\varphi$ were constant onto the boundary (i.e. $\mathbf{t}\cdot\underline{\nabla}\varphi=0$ on $\partial\Omega$), the orthogonality would result verified with $\mathbf{n}\cdot\mathbf{w}_2\neq 0$. Observe that the orthogonality would be verified with suitable periodical boundary conditions, too.

It is worthwhile distinguishing the concept of orthogonality used in the above context from that one, which is given in $\Re^3$ in the sense of *standard inner product* of two vectors defined as $\mathbf{x}^{\mathrm{T}}\cdot\mathbf{y}=\sum_{i=1}^n x_i y_i\in\Re$. Based on this latter, two vectors are orthogonal in $\Re^3$ in the sense of perpendicularity embodied in the Pythagorean theorem that states they are orthogonal if and only if $\|\mathbf{x}\|^2+\|\mathbf{y}\|^2=\|\mathbf{x}-\mathbf{y}\|^2$ implying $\mathbf{x}^{\mathrm{T}}\cdot\mathbf{y}=0$. Such definition naturally extends to a higher dimensions space $\Re^n$.

[¶]The divergence and the gradient are adjoint operators: $\int_\Omega \mathbf{w}_2\cdot\underline{\nabla}\varphi\,\mathrm{d}V=-\int_\Omega\varphi\underline{\nabla}\cdot\mathbf{w}_2\,\mathrm{d}V$ if $\mathbf{n}\cdot\mathbf{w}_2=0$ on the boundary.

The *existence* of the decomposition, when the boundary condition $(2)_4$ is prescribed, results from the fact that the problem constituted by the Poisson Equation $(2)_2$ associated to the boundary condition $(2)_4$

$$\begin{aligned} \nabla^2\varphi &= \rho, \quad \mathbf{x}\in\Omega \\ \mathbf{n}\cdot\underline{\nabla}\varphi &= w_n, \quad \mathbf{x}\in\partial\Omega \end{aligned} \tag{4}$$

admits a unique (apart from a constant) solution $\varphi$ because the required compatibility condition (see Reference [30]):

$$\int_\Omega \rho\,\mathrm{d}V = \int_{\partial\Omega} w_n\,\mathrm{d}S \Rightarrow \int_\Omega \underline{\nabla}\cdot\underline{\nabla}\varphi\,\mathrm{d}V = \int_{\partial\Omega}\mathbf{n}\cdot\underline{\nabla}\varphi\,\mathrm{d}S = \int_{\partial\Omega} w_n\,\mathrm{d}S \tag{5}$$

is verified by the Neumann boundary condition $(4)_2$, which implies that $\mathbf{w}_2$ must be parallel to $\partial\Omega$ (but without prescribing its tangential vector component).

*Remark 1*
The mathematical problem constituted by the Poisson equation $(4)_1$, associated to non-homogeneous Neumann boundary conditions $(4)_2$, is *equivalent* to that one with prescribed homogeneous Neumann boundary conditions[||] and a modified source term $\tilde{\rho}$ obtained when the divergence operator is defined onto the subspace of vectors with vanishing normal component on the boundary $\partial\Omega$.

Analogously, the solution of the problem constituted by the Poisson Equation $(2)_3$ (being $\underline{\nabla}\wedge(\underline{\nabla}\wedge\mathbf{b}) = -\nabla^2\mathbf{b}$) associated to the boundary condition $(2)_5$

$$\begin{aligned} \nabla^2\mathbf{b} &= -\mathbf{r}, \quad \mathbf{x}\in\Omega \\ \mathbf{n}\wedge(\underline{\nabla}\wedge\mathbf{b}) &= \mathbf{w}_t, \quad \mathbf{x}\in\partial\Omega \end{aligned} \tag{6}$$

exists and is unique because the compatibility condition:

$$\int_S \mathbf{n}\cdot\mathbf{r}\,\mathrm{d}S = \int_{\partial S}\mathbf{w}_t\cdot\mathrm{d}\mathbf{l} \tag{7}$$

being $S$ the part of $\partial\Omega$ spanned by the contour $\partial S$ and $\mathbf{l}$ the tangential unit vector to $\partial S$, is verified by the Dirichlet boundary condition $(6)_2$ which implies that $\mathbf{w}_1$ (i.e. $\underline{\nabla}\varphi$) must be parallel to the unit vector outward to $\partial\Omega$ (its tangential components are null i.e. $\mathbf{t}_1\cdot\underline{\nabla}\varphi = \mathbf{t}_2\cdot\underline{\nabla}\varphi = 0$). As a consequence, only the tangential vector $\mathbf{w}_{2t}$ is prescribed without forcing a vanishing (or a prescribed value) normal component of $\mathbf{w}_2$.

The *uniqueness* of the HHD is proved by supposing that there exist two different orthogonal decompositions $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 = \underline{\nabla}\varphi + \underline{\nabla}\wedge\mathbf{b}$ and $\mathbf{w} = \mathbf{w}_1' + \mathbf{w}_2' = \underline{\nabla}\varphi' + \underline{\nabla}\wedge\mathbf{b}'$ both having on $\partial\Omega$ either $\mathbf{n}\cdot\mathbf{w}_2 = \mathbf{n}\cdot\mathbf{w}_2' = 0$ or $\mathbf{n}\wedge\mathbf{w}_1 = \mathbf{n}\wedge\mathbf{w}_1' = \mathbf{0}$ in case of the problem (4) or (6), respectively.

[||] In fact, sometimes the solution of the pressure equation is referred to as that of a homogeneous Neumann problem.

It results $\mathbf{0}=\underline{\nabla}(\varphi-\varphi')+\mathbf{w}_2-\mathbf{w}_2'$ from which, by taking the inner product with $\mathbf{w}_2-\mathbf{w}_2'$, one has:

$$\begin{aligned}
0 &= \int_\Omega [(\mathbf{w}_2-\mathbf{w}_2')\cdot(\mathbf{w}_2-\mathbf{w}_2')+(\mathbf{w}_2-\mathbf{w}_2')\cdot\underline{\nabla}(\varphi-\varphi')]\,\mathrm{d}V \\
&= \int_\Omega [(\mathbf{w}_2-\mathbf{w}_2')\cdot(\mathbf{w}_2-\mathbf{w}_2')]\,\mathrm{d}V - \int_\Omega (\mathbf{w}_2\cdot\underline{\nabla}\varphi'+\mathbf{w}_2'\cdot\underline{\nabla}\varphi)\,\mathrm{d}V \\
&= \int_\Omega \|\mathbf{w}_2-\mathbf{w}_2'\|^2\,\mathrm{d}V - \int_\Omega [\underline{\nabla}\cdot(\mathbf{w}_2\varphi')-\varphi'\underline{\nabla}\cdot\mathbf{w}_2]\,\mathrm{d}V - \int_\Omega [\underline{\nabla}\cdot(\mathbf{w}_2'\varphi)-\varphi\underline{\nabla}\cdot\mathbf{w}_2']\,\mathrm{d}V \\
&= \int_\Omega \|\mathbf{w}_2-\mathbf{w}_2'\|^2\,\mathrm{d}V - \int_{\partial\Omega} \mathbf{n}\cdot\mathbf{w}_2\varphi'\,\mathrm{d}S - \int_{\partial\Omega} \mathbf{n}\cdot\mathbf{w}_2'\varphi\,\mathrm{d}S \\
&= \int_\Omega \|\mathbf{w}_2-\mathbf{w}_2'\|^2\,\mathrm{d}V
\end{aligned}$$

or, by taking the inner product with $\mathbf{w}_1-\mathbf{w}_1'$, one gets

$$\begin{aligned}
0 &= \int_\Omega [(\mathbf{w}_1-\mathbf{w}_1')\cdot(\mathbf{w}_1-\mathbf{w}_1')+(\mathbf{w}_1-\mathbf{w}_1')\cdot(\mathbf{w}_2-\mathbf{w}_2')]\,\mathrm{d}V \\
&= \int_\Omega [(\mathbf{w}_1-\mathbf{w}_1')\cdot(\mathbf{w}_1-\mathbf{w}_1')]\,\mathrm{d}V - \int_\Omega (\mathbf{w}_2\cdot\underline{\nabla}\varphi'+\mathbf{w}_2'\cdot\underline{\nabla}\varphi)\,\mathrm{d}V \\
&= \int_\Omega \|\mathbf{w}_1-\mathbf{w}_1'\|^2\,\mathrm{d}V - \int_\Omega [\underline{\nabla}\cdot(\mathbf{b}\wedge\underline{\nabla}\varphi')+\underline{\nabla}\cdot(\mathbf{b}'\wedge\underline{\nabla}\varphi)]\,\mathrm{d}V \\
&= \int_\Omega \|\mathbf{w}_1-\mathbf{w}_1'\|^2\,\mathrm{d}V - \int_{\partial\Omega} \mathbf{n}\cdot(\mathbf{b}\wedge\underline{\nabla}\varphi')\,\mathrm{d}S - \int_{\partial\Omega} \mathbf{n}\cdot(\mathbf{b}'\wedge\underline{\nabla}\varphi)\,\mathrm{d}S \\
&= \int_\Omega \|\mathbf{w}_1-\mathbf{w}_1'\|^2\,\mathrm{d}V - \int_{\partial\Omega} \mathbf{b}\cdot(\underline{\nabla}\varphi'\wedge\mathbf{n})\,\mathrm{d}S - \int_{\partial\Omega} \mathbf{b}'\cdot(\underline{\nabla}\varphi\wedge\mathbf{n})\,\mathrm{d}S \\
&= \int_\Omega \|\mathbf{w}_1-\mathbf{w}_1'\|^2\,\mathrm{d}V
\end{aligned} \tag{8}$$

therefore, in the first case the integral vanishes if $\mathbf{w}_2=\mathbf{w}_2'$ (positivity of the integrand) and consequently $\underline{\nabla}\varphi=\underline{\nabla}\varphi'$ whilst in the second case if $\underline{\nabla}\varphi=\underline{\nabla}\varphi'$ and thus $\mathbf{w}_2=\mathbf{w}_2'$.

Observe also that for the uniqueness of the orthogonal decomposition, the boundary conditions $\mathbf{n}\cdot\mathbf{w}_2=\mathbf{n}\cdot\mathbf{w}_2'=0$ as well as $\mathbf{n}\wedge\mathbf{w}_1=\mathbf{n}\wedge\mathbf{w}_1'=\mathbf{0}$ on $\partial\Omega$ are sufficient but not necessary. For example, one could have $\mathbf{n}\cdot\mathbf{w}_2\neq 0$ and $\mathbf{n}\cdot\mathbf{w}_2'\neq 0$ and still retain a unique decomposition if $\varphi$ and $\varphi'$ were constants on $\partial\Omega$.

## 3. THE APPLICATION OF THE HHD THEOREM FOR CONTINUOUS OPERATOR: THE DETERMINATION OF THE EULERIAN ACCELERATION

In solving the system of differential equations for incompressible flows, the velocity vector and the pressure gradient fields result coupled each other by the continuity constraint, because the pressure is only a Lagrange multiplier, not a thermodynamic state variable. Since this coupling leads to computational procedures for solving continuity and momentum equations in a Stokes-like system, splitting methodologies are often implemented. Owing to their simplicity in de-coupling the problem and solving separately the parabolic/elliptic equations, projection methods are extensively used. Hence, the application of the HHD theorem into a projection method for solving the incompressible form of the Navier–Stokes equations is now discussed and its implications analysed. In particular, the time-continuous formulation of the momentum equation will be now considered. Let us remark that the mathematical problem will consist in determining the unique orthogonal decomposition of an *assigned or computed* vector field $\mathbf{w}$, not in determining it by solving Equation (1).

Therefore, the vector field $\mathbf{w}$ will be associated to that part of the momentum equation, which can be computed from the only initial velocity field. Specifically, the advective and diffusive terms are known while the Eulerian acceleration and the pressure gradient represent the unknown terms of the decomposition. The goal of this section is to illustrate the implications of the HHD theorem when one has:

- To determine the Eulerian acceleration $\mathbf{a}(\mathbf{x},t_0)$ from the knowledge of initial and boundary conditions prescribed in terms of the velocity vector field $\mathbf{v}(\mathbf{x},t_0)$,
- Then, to solve the initial value problem $\partial\mathbf{v}/\partial t=\mathbf{a}$ along with the initial condition $\mathbf{a}(\mathbf{x},t_0)=\mathbf{a}_0(\mathbf{x})$ and obtain the desired solution in terms of the divergence-free velocity field $\mathbf{v}(\mathbf{x},T)$ along with the pressure gradient.

Typically, standard projection methods work in terms of the velocity field and start from a different point of view (e.g. see Reference [14] for a review) as the HHD is exploited only after that a second-order time-accurate integration (with the AB/CN scheme) has been performed in order for the vector field to be computed and thereafter decomposed. This other procedure is addressed in the next section so that the differences between the boundary condition can be highlighted for both procedures.

Hence, from the momentum equation, define in $\Omega'=\Omega\times(t_0,T)$ the Eulerian acceleration field $\mathbf{a}$

$$\mathbf{a}(\mathbf{x},t):=\frac{\partial\mathbf{v}}{\partial t}=-\underline{\nabla}\cdot(\mathbf{v}\mathbf{v})+\underline{\nabla}\cdot\left(\frac{1}{Re}\,\underline{\nabla}\mathbf{v}\right)-\underline{\nabla}p'$$

being $\mathbf{a}$ divergence-free owing to the continuity constraint $\underline{\nabla}\cdot\mathbf{v}=0$, $Re$ the Reynolds number and having denoted by $p'$ the non-dimensional pressure.

Assume that the initial and boundary conditions are prescribed in terms of the velocity field $\mathbf{v}$ while the initial pressure gradient is unknown. Furthermore, assume that the *compatibility conditions* for the Navier–Stokes equations, i.e. the necessary and sufficient conditions on the initial data on $\partial\Omega$ at $t_0$ are fulfilled. Thus, regularity results of the strong solution $\mathbf{v}(\mathbf{x},t_0)$ are ensured in the bounded domain (for more details, see Temam [31]).

Now, consider the vector field to be decomposed, say $\mathbf{a}^*(\mathbf{x},t)$, provided by the convection and diffusion of $\mathbf{v}$, i.e.:

$$\mathbf{a}^*:=-\underline{\nabla}\cdot(\mathbf{v}\mathbf{v})+\underline{\nabla}\cdot\left(\frac{1}{Re}\,\underline{\nabla}\mathbf{v}\right)$$

so, it results known from the prescribed initial data. Thus, the momentum equation is simply recast as

$$\mathbf{a}^* = \mathbf{a} + \underline{\nabla} p' \tag{9}$$

The attractive feature of Equation (9) stands in the fact that it appears in a form which accords to the decomposition $(2)_1$, wherein $\mathbf{a}^*$ takes the role of $\mathbf{w}, \mathbf{a}$ that of $\mathbf{w}_2$ and $\underline{\nabla} p'$ that of $\mathbf{w}_1$. However, in order for Equation (9) to be the HHD of $\mathbf{a}^*$, the vector fields must satisfy Equations (2), now rewritten as

$$\begin{aligned}
&\underline{\nabla} \cdot \underline{\nabla} p' = \underline{\nabla} \cdot \mathbf{a}^* \\
&\underline{\nabla} \wedge \mathbf{a} = \underline{\nabla} \wedge \mathbf{a}^*, \quad \mathbf{x} \in \Omega \\
&\text{with} \\
&\mathbf{n} \cdot \mathbf{a}^* = a_n^* \\
&\text{or:} \\
&\mathbf{n} \wedge \mathbf{a}^* = \mathbf{a}_t^*, \quad \mathbf{x} \in \partial\Omega
\end{aligned} \tag{10}$$

Again, Equation (10) highlight that, for the present purpose, the vector field $\mathbf{a}^*$ at time $t_0$, its divergence and curl are assumed known, so that the problem can be stated as follows

*Problem*

For any vector field $\mathbf{a}^*(\mathbf{x}, t_0) = \mathbf{a}_0^*(\mathbf{x})$, determined from known initial data $\mathbf{v}(\mathbf{x}, t_0)$ in $\mathbf{x} \in \Omega$ and boundary conditions $\mathbf{v}_\partial(\mathbf{x}, t_0)$ on $\mathbf{x} \in \partial\Omega$ (i.e. satisfying both normal and tangential physical values), determine its unique orthogonal decomposition.

Hence, one wonders if the two vector fields $(\mathbf{a}, \underline{\nabla} p')$ express or not such a decomposition. When the HHD is performed, one can solve the initial value problem $\partial\mathbf{v}/\partial t = \mathbf{a}$ along with the initial condition $\mathbf{a}(\mathbf{x}, t_0) = \mathbf{a}_0(\mathbf{x})$ and expresses the velocity field $\mathbf{v}(\mathbf{x}, T)$, for example according to:

$$\mathbf{v}(\mathbf{x}, T) = \mathbf{v}(\mathbf{x}, t_0) + \int_{t_0}^{T} \mathbf{a}\, \mathrm{d}t = \mathbf{v}(\mathbf{x}, t_0) + (T - t_0)\mathbf{a}_0 + \frac{(T - t_0)^2}{2} \left.\frac{\partial \mathbf{a}}{\partial t}\right|_{t_0} + \cdots, \quad \mathbf{x} \in \Omega \tag{11}$$

Formally, the procedure is now closed, as the updated velocity field (11) that should satisfy the continuity constraint, provides the non-solenoidal acceleration field $\mathbf{a}^*(\mathbf{x}, T)$, which is suitable to be next decomposed as $\mathbf{a}^*(\mathbf{x}, T) = \mathbf{a}(\mathbf{x}, T) + \underline{\nabla} p'(\mathbf{x}, T)$. However, as the HHD is mathematically well posed when only one boundary condition is prescribed, one wonders what properties will be ensured by the obtained boundary expression $\mathbf{v}_\partial(\mathbf{x}, T)$ and consequently by $\mathbf{a}_\partial(\mathbf{x}, T)$. The accuracy of a projection method will depend on the way whereby the splitting between acceleration and pressure gradient causes an error that has some component along the velocity vector field. This fact suggests that the orthogonality of the decomposition is important.

Let us now consider some different types of boundary conditions dictated by the real physical problem to be solved.

### *3.1. Case I: decomposition* (9) *with* $\mathbf{n}\cdot\mathbf{a}=0$ *on* $\partial\Omega$

This case is accomplished, for example, by flows totally confined by not permeable walls (e.g.: lid-driven or natural convection in cavity), steady injection/suction along the walls as well as by channel flows with prescribed inflow and outflow steady conditions. The consequent initial and boundary conditions, prescribed in terms of $\mathbf{v}$, allow us obtaining the acceleration $\mathbf{a}^*(\mathbf{x},t_0)=-\underline{\nabla}\cdot(\mathbf{v}_0\mathbf{v}_0)+1/Re\,\nabla^2\mathbf{v}_0$ and decompose it.

By substituting $\mathbf{w}=\mathbf{a}^*$, $\mathbf{w}_1=\underline{\nabla}p'$ and $\mathbf{w}_2=\mathbf{a}$ in the demonstrations (1)–(8), one directly sees that the momentum equation (9) already expresses the unique decomposition of the vector field $\mathbf{a}^*$ in $\Omega$. Therefore, it is possible to introduce the *true* orthogonal projection operator** $P_H$ that is symmetric $P_H=(P_H)^{\mathrm{T}}$, bounded $||P_H||=1$ and $||P_H(\mathbf{a}^*)||\leqslant||\mathbf{a}^*||$, idempotent $(P_H)^2=P_H$ (being $P_H[P_H(\mathbf{a}^*)]=\mathbf{a}$, for the uniqueness of the decomposition). It extracts the divergence-free part of $\mathbf{a}^*$ which is parallel to the boundary: by applying $P_H$ onto Equation (9) and taking into account that $P_H(\underline{\nabla}p')=\mathbf{0}$, $P_H(\mathbf{a})=\mathbf{a}$ one gets $P_H(\mathbf{a}^*)=\mathbf{a}$. In conclusion, the boundary condition $\mathbf{n}\cdot\mathbf{a}=0$ is sufficient to ensure that (9) expresses the orthogonal decomposition of $\mathbf{a}^*$ while also ensuring the existence and uniqueness of the pressure gradient $\underline{\nabla}p'=\mathbf{a}^*-\mathbf{a}=(I-P_H)(\mathbf{a}^*)$. This pressure gradient is obtained by solving the Poisson equation $(10)_1$ at $t_0$, $\nabla^2p'=\underline{\nabla}\cdot\mathbf{a}^*(\mathbf{x},t_0)$ along with the inhomogeneous Neumann boundary condition $\partial p'/\partial n=a_n^*(\mathbf{x},t_0)$ prescribed on $\mathbf{x}\in\partial\Omega$. Let us remind that Remark 1 explains how, alternatively, this pressure problem can be associated to homogeneous Neumann boundary condition $\partial p'/\partial n=0$ provided that the source term is modified. Some further observations are concerned about the decomposition onto the boundary. According to the standard inner product, perpendicularity requires $\|\mathbf{a}\|^2+\|\underline{\nabla}p'\|^2=\|\mathbf{a}-\underline{\nabla}p'\|^2$ onto the boundary. Actually one gets $\mathbf{a}\cdot\underline{\nabla}p'|_{\partial\Omega}=(a_{t_1}(\partial p'/\partial t_1)+a_{t_2}(\partial p'/\partial t_2)|_{\partial\Omega}$, $t_1$ and $t_2$ representing the tangential directions of a local reference system, therefore the two vectors result perpendicular only for some specific flow conditions.

Finally, the Eulerian acceleration is determined from (9) as $\mathbf{a}_0=\mathbf{a}_0^*-\underline{\nabla}p'|_{t_0}$ and according to Equation (11), the time integration can be performed up to first-order terms $\mathbf{v}(\mathbf{x},T)=\mathbf{v}(\mathbf{x},t_0)+\Delta t\,\mathbf{a}_0$. Higher order accurate integrations can be performed provided that derivatives of $\mathbf{a}$ are expressed by means of (9) in a Lax–Wendroff integration type [15–18]. For example, up to $O(\Delta t^2)$ terms, one can consider a second HHD $\partial\mathbf{a}/\partial t|_{t_0}=\partial\mathbf{a}^*/\partial t|_{t_0}-\underline{\nabla}\partial p'/\partial t|_{t_0}$, provided that $\mathbf{n}\cdot\partial\mathbf{a}/\partial t|_{t_0}=0$, wherein $\partial\mathbf{a}^*/\partial t|_{t_0}=-\underline{\nabla}\cdot(\mathbf{a}_0\mathbf{v}_0+\mathbf{v}_0\mathbf{a}_0)+(1/Re)\nabla^2\mathbf{a}_0$ can be obtained from the previous vector fields and the gradient of the pressure derivative is obtained by solving $\nabla^2(\partial p'/\partial t)|_{t_0}=\underline{\nabla}\cdot(\partial\mathbf{a}^*/\partial t)|_{t_0}$ with proper boundary conditions. As regards with the consequent boundary conditions, only the normal component is exactly imposed. On the other side, it is well known that the fulfilment of the tangential component remains fairly accurate [14, 23]. Although the field $\mathbf{a}_0^*$ is constructed from the velocity field $\mathbf{v}_0$, having a correct tangential assignment all the way up to the boundary, one has $\mathbf{n}\wedge(\mathbf{a}_0+\underline{\nabla}p'|_{t_0})=\mathbf{a}_{t,0}^*$ and the tangential component of $\mathbf{a}$ will not necessarily satisfy the flow condition in effect. What must be ensured is that the field remains smooth close to the boundary and the tangential component is as accurate as the field into the interior.

**In the real space it can be shown that $\mathbf{a}=P_H(\mathbf{a}^*)=\int_{R^3}\mathbf{K}(\mathbf{x}'-\mathbf{x})\cdot\mathbf{a}^*(\mathbf{x}')\,\mathrm{d}\mathbf{x}'$ being $\mathbf{K}(\mathbf{x})=\mathbf{I}\delta(\mathbf{x})-\frac{1}{4\pi}\left(\frac{\mathbf{I}}{|\mathbf{x}|^3}-3\frac{\mathbf{x}\mathbf{x}^{\mathrm{T}}}{|\mathbf{x}|^5}\right)$.

### 3.2. *Case II: decomposition* (9) *with* $\mathbf{n}\cdot\mathbf{a}\neq 0$ *on* $\partial\Omega$

This case is accomplished, for example, by confined flows with unsteady injection/suction or with unsteady inflow/outflow (e.g. turbulent channel or backward facing-step flows), periodical flows (homogeneous turbulent flows, exact solutions) or other related problems. Again, initial and boundary conditions allow us to obtain the acceleration $\mathbf{a}^*$ and decompose it.

When assigning $\mathbf{n}\cdot\mathbf{a}=a_n\neq 0$, that makes the demonstrations (1)–(8) no longer straightforwardly applicable to (9). Nevertheless, in order for the decomposition (9) to be still orthogonal, it would be sufficient to subsist that condition $\int_{\partial\Omega}\mathbf{n}\cdot\mathbf{a}\,p'\,\mathrm{d}S=0$ subsists (this happens to be true, as we shall see, with suitable periodical boundary conditions as well as if $p'$ is constant along $\partial\Omega$), whereas the uniqueness of such decomposition would remain undetermined (see Equation (8)). However, one can look for the unique orthogonal decomposition of $\mathbf{a}^*$ and reconsider the previous problem in terms of the three differential forms of Hodge decomposition that mathematically expresses as

$$\mathbf{a}^* = \mathbf{a}' + \underline{\nabla}\phi \tag{12}$$

being $\mathbf{a}'$ a divergence-free vector field parallel to $\partial\Omega$. As the momentum equation (9) still applies, one gets also:

$$\mathbf{a} = \mathbf{a}' + \underline{\nabla}\phi - \underline{\nabla}p' \equiv \mathbf{a}' + \underline{\nabla}f \tag{13}$$

which is the desired relation between the Eulerian acceleration and the *true* orthogonal projection of $\mathbf{a}^*$, i.e. $P_H(\mathbf{a}^*)=\mathbf{a}'$, onto the subspace $H$ of the divergence-free vectors parallel to the boundary.

The existence and uniqueness of the vector field $\underline{\nabla}f=\underline{\nabla}\phi-\underline{\nabla}p'$ is ensured since, being $\mathbf{a}$ divergence-free, one gets from Equation (13) the problem constituted by the Laplace equation along with non-homogeneous Neumann boundary conditions

$$\begin{aligned} \nabla^2 f &= 0, \quad \mathbf{x}\in\Omega \\ \mathbf{n}\cdot\underline{\nabla}f &= a_n, \quad \mathbf{x}\in\partial\Omega \end{aligned} \tag{14}$$

which admits a unique solution $f$ (apart from a constant) because the necessary compatibility condition

$$\int_\Omega \nabla^2 f\,\mathrm{d}V = \int_{\partial\Omega}\mathbf{n}\cdot\underline{\nabla}f\,\mathrm{d}S = \int_{\partial\Omega} a_n\,\mathrm{d}S = 0 \tag{15}$$

is satisfied. Therefore, the resulting difference vector between $\mathbf{a}$ and $\mathbf{a}'$, expressed by Equation (13), is given by a gradient of a harmonic function. This result is consistent to the original theory of Hodge.

Now, the counterpart of the decomposition addressed in Case I is performed by applying the projector $P_H$ on Equation (12), namely $P_H(\mathbf{a}^*)=\mathbf{a}'$ and obtaining the unique gradient field $\underline{\nabla}\phi=\mathbf{a}^*-\mathbf{a}'=(I-P_H)(\mathbf{a}^*)$. In practice, one solves at $t_0$ first the problem constituted by the Poisson equation along with the Neumann boundary condition:

$$\begin{aligned} \nabla^2\phi &= \underline{\nabla}\cdot\mathbf{a}_0^*, \quad \mathbf{x}\in\Omega \\ \mathbf{n}\cdot\underline{\nabla}\phi &= \mathbf{n}\cdot\mathbf{a}_0^*, \quad \mathbf{x}\in\partial\Omega \end{aligned} \tag{16}$$

and obtains the field $\mathbf{a}'_0 = \mathbf{a}^*_0 - \underline{\nabla}\phi|_{t_0}$, then the harmonic function $f$ is obtained by solving the problem (14) so that, finally, one gets the Eulerian acceleration according to $\mathbf{a}_0 = \mathbf{a}'_0 + \underline{\nabla} f|_{t_0}$.

This way, one proceeds by performing two successive decompositions along directions dictated by the orthogonality criterion: first the HHD of $\mathbf{a}^*$ allows us to obtain the field $\mathbf{a}'$ by exploiting (12), then Equation (13) can be considered as a further HHD this time, being the divergence-free vector $\mathbf{a}'$ prescribed and the vector field $\mathbf{a}$ to be determined.

One wonders if this is the only way to get $\mathbf{a}$ as well as if an alternative way provides the same result in terms of final accuracy. As a matter of fact, it can be shown that the Eulerian acceleration can be directly determined in one step. To this aim, first, let us introduce the definition of an *oblique projector* i.e. an operator denoted by

⇒ $P_X$ onto $X$ such that $P_X(\mathbf{a}^*) = \mathbf{a}$, being $\mathbf{a}$ defined in the divergence-free space $X = \{\mathbf{v} \in \mathbf{L}^2(\Omega)\colon\ \underline{\nabla}\cdot\mathbf{a} = 0,\ \mathbf{n}\cdot\mathbf{a} = a_n \text{ on } \partial\Omega\}$ if the normal component is assigned.

The fact that one has a divergence-free vector field, which is not parallel to the boundary, introduces a new issue on uniqueness. The divergence and the gradient are not adjoint operators although $P_X$ is still idempotent. On the basis of the previous analyses, depending on the type of boundary conditions, the vectors $(\mathbf{a}, \underline{\nabla} p')$ could still result orthogonal, in the sense given by Equations (3), and $P_X$ could be still an orthogonal projector but they would not necessarily represent the unique decomposition of $\mathbf{a}^*$. This definition of oblique projection $P_X$ highlights the fact that projection is neither in $H$ nor in $H'$, more in general, $\mathbf{a}$ and $\underline{\nabla} p'$ are vectors belonging to non-orthogonal subspaces, i.e. $\int_{\partial\Omega} \mathbf{n}\cdot\mathbf{a}\, p'\, \mathrm{d}S \neq 0$. In this sense, as also observed in Reference [20], the HHD principle must be reconsidered in a more general sense, as well as Equation (9) must be reconsidered as a decomposition along two general directions.

Therefore, for the given vector field $\mathbf{a}^*_0(\mathbf{x})$, one proceeds in doing the decomposition (9) by computing the gradient $\underline{\nabla} p' = (I - P_X)(\mathbf{a}^*)$ by solving the problem constituted by the Poisson equation with non-homogeneous Neumann boundary conditions:

$$\begin{aligned} \nabla^2 p' &= \underline{\nabla}\cdot\mathbf{a}^*, \quad \mathbf{x}\in\Omega \\ \mathbf{n}\cdot\underline{\nabla} p' &= \mathbf{n}\cdot\mathbf{a}^* - a_n, \quad \mathbf{x}\in\partial\Omega \end{aligned} \tag{17}$$

existence and uniqueness (always apart from a constant) of $p'$, being ensured from the compatibility condition, fulfilled by the continuity constraint.

Let us observe that Remark 1 extends also to the solution of problem (17) stating its equivalence with a Poisson problem, associated to homogeneous Neumann boundary conditions, having a modified source term, obtained when the divergence operator is defined onto the subspace of vector function with normal component equal to $a_n$.

### *3.3. Case III: decomposition* (9) *with* $\mathbf{n}\wedge\underline{\nabla} p' = 0$ *on* $\partial\Omega$

This case prescribes the boundary condition $(10)_4$ thus, $\mathbf{t}\cdot\underline{\nabla} p'|_{\partial\Omega} = 0$ (the case of flows having constant pressure along the boundary of the domain) and the pressure gradient has the only component directed along the normal to the boundary.

Again, by considering Equations (1)–(8) with $\mathbf{w} = \mathbf{a}^*$, $\mathbf{w}_1 = \underline{\nabla} p'$ and $\mathbf{w}_2 = \mathbf{a} = (\underline{\nabla}\wedge\mathbf{b})$, one sees that $\mathbf{n}\wedge\underline{\nabla} p' = \mathbf{0}$ is the sufficient condition to have (9) directly expressing the unique HHD of $\mathbf{a}^*$ in $\Omega$. Hence, it is possible to introduce the orthogonal projection operator $P_{H'}$ such that it extracts the divergence-free part of $\mathbf{a}^*$, its tangential vector component $(10)_4$ being assigned on the boundary.

Initial and boundary conditions allow to determine the vectors $\mathbf{a}_0^*(\mathbf{x})$ and $\mathbf{r}_0(\mathbf{x}) = \underline{\nabla} \wedge \mathbf{a}_0^*(\mathbf{x})$ and, in order for $\mathbf{a}_0^*(\mathbf{x})$ to be projected onto the subspace $H'$, one proceeds by applying $P_{H'}$ on $\mathbf{a}_0^*(\mathbf{x})$, taking into account that $P_{H'}(\underline{\nabla} p') = \mathbf{0}$, $P_{H'}(\mathbf{a}) = \mathbf{a}$, and solving $P_{H'}(\mathbf{a}^*) = \mathbf{a}$. The potential divergence-free vector $\mathbf{b}$ is computed by solving the problem constituted by the Poisson equation along with Dirichlet boundary conditions:

$$\begin{aligned} \nabla^2 \mathbf{b} &= -\mathbf{r}_0, \quad \mathbf{x} \in \Omega \\ \mathbf{n} \wedge (\underline{\nabla} \wedge \mathbf{b}) &= \mathbf{n} \wedge \mathbf{a}_0^*, \quad \mathbf{x} \in \partial\Omega \end{aligned} \tag{18}$$

the compatibility condition ensuring the existence and uniqueness of $\mathbf{b}$. This field allows us to determine the Eulerian acceleration $\mathbf{a}_0(\mathbf{x}) = (\underline{\nabla} \wedge \mathbf{b})|_{t_0}$ and then the velocity from Equation (11).

Equivalently, as the curl of $\mathbf{a}_0^*(\mathbf{x})$ coincides with the local time derivative of the vorticity $\zeta$ at $t_0$:

$$\left.\frac{\partial \boldsymbol{\zeta}}{\partial t}\right|_{t_0} := \mathbf{r}_0(\mathbf{x}) \tag{19}$$

one gets (e.g. see References [32, 33]) the well-known *potential vector-vorticity formulation* that is based on the solution of the initial value problem $\partial\boldsymbol{\zeta}/\partial t = \mathbf{r}$ with initial data $\boldsymbol{\zeta}_0(\mathbf{x}) = \underline{\nabla} \wedge \mathbf{v}_0(\mathbf{x})$ and proper boundary conditions:

$$\boldsymbol{\zeta}(\mathbf{x}, T) = \boldsymbol{\zeta}(\mathbf{x}, t_0) + \int_{t_0}^{T} \underline{\nabla} \wedge \mathbf{a}^* \, \mathrm{d}t = \boldsymbol{\zeta}(\mathbf{x}, t_0) + (T - t_0)\mathbf{r}_0 + \frac{(T - t_0)^2}{2} \left.\frac{\partial \mathbf{r}}{\partial t}\right|_{t_0} + \cdots, \quad \mathbf{x} \in \Omega \tag{20}$$

from which one restates the problem (18) integrated in time. Now, the vector $\mathbf{n} \wedge \mathbf{v}(\mathbf{x}, T)$ can be determined on the boundary, because of the $(18)_2$, but the normal component value does not result explicitly fulfilled from the HHD. In Reference [32] it is reported a strategy to face this issue in case of 2D multi-connected domains. In the framework of such formulation, spurious vorticity, or vorticity errors, were removed on the basis of the HHD theorem [34]. By taking the curl of (18), analogous considerations lead to the *velocity-vorticity formulation*.

The case $\mathbf{n} \wedge \underline{\nabla} p' \neq \mathbf{0}$ on $\partial\Omega$, is not analysed because, analogously to the Case II, one can determine a second potential vector in order to report us to the Case III and determine the two unique orthogonal components of the HHD.

## 4. THE APPLICATION OF THE HHD THEOREM FOR TIME-DISCRETIZED OPERATORS: THE SECOND-ORDER TIME ACCURATE *PRESSURE-FREE PROJECTION METHOD*

In many studies, the implementation of projection methods is performed in a quite different way from that one illustrated in the previous section. More specifically, the HHD applies for determining a divergence-free velocity field after that an intermediate non-solenoidal velocity, say $\mathbf{v}^*$, is computed. Among the others, one of the most common projection methods is the second-order FTSM by Kim and Moin [10] (so-called *pressure-free projection* (PFP) method, e.g. see Reference [14]). The intermediate velocity satisfies a semi-implicit discrete equation, which is obtained by applying the second-order AB/CN time integration onto the momentum equation (9) while disregarding the pressure terms. Next, the velocity $\mathbf{v}^*$ is decomposed in order for the velocity $\mathbf{v}(T)$ to be the divergence-free. As such a projection method is still based on the application of the HHD theorem and is usually adopted for simulating flows with prescribed normal component $\mathbf{n}\cdot\mathbf{v}_\partial \neq 0$, the discussions addressed in Cases I and II have to be considered as guidelines for applying the decomposition criterion.

In order for the orthogonal projector $P_H$ to be defined some differences appear whereas the vector to be decomposed is the provisional velocity field $\mathbf{v}^*$ instead of being $\mathbf{a}^*$. In fact, one has that $\mathbf{n}\cdot\mathbf{v}_\partial = 0$ could be not in effect for the studied flow problem, whereas $\mathbf{n}\cdot\mathbf{a}_\partial = 0$ is a flow condition which could be more generally accomplished. Such a case is encountered for example in the steady backward facing-step flows. Nevertheless, as discussed in Section 3.1–2, for performing an orthogonal decomposition (not unique) it could not be necessarily a null velocity normal component.

Hitherto, one remarks:

- In the PFP method, the provisional vector field $\mathbf{v}^*$, has no meaning of a time-continuous function but it is just a mathematical position into the time-discretized prediction equation. This means that the field $\mathbf{a}^*$, previously introduced, must not be considered representative of the time derivative of $\mathbf{v}^*$. This aspect is fundamental in deriving consistent boundary conditions for the closure of the fractional-based equations [35, 36].
- Several modal analyses have demonstrated that when the AB/CN-based prediction equation is associated to the intermediate boundary conditions proposed in [10] a numerical boundary layer is generated [14, 20, 23–27]. Nevertheless, although created by inconsistent boundary conditions, when such boundary layer mode is orthogonal to the space of divergence-free vector fields, the projected velocity field does not contain such errors and full second-order accuracy is retained all the way up to the boundary. Thus, the actual accuracy of the PFP method when based on the oblique projector $P_X$, deserves a careful consideration.
- The time integration is performed within a prediction step, namely before of performing the decomposition. In doing so, the meaning of the gradient field, which appears into the decomposition, is altered. In fact, such a field is an auxiliary variable and it will be only an approximation of the real pressure gradient.

Now the PFP method for prescribed non-homogeneous Dirichlet conditions is briefly addressed. For prescribed divergence-free vector fields, let us say $\mathbf{v}(\mathbf{x},t^n)=\mathbf{v}^n$ and $\mathbf{v}(\mathbf{x},t^{n-1})=\mathbf{v}^{n-1}$, $t^{n-1}=t^n-\Delta t$, one integrates Equation (9) in the time interval $[t^n, t^{n+1}=t^n+\Delta t]$, according to the CN scheme for the diffusive term and the AB scheme for the advective ones. After having disregarded the integral pressure term, the provisional field $\mathbf{v}^*$ is defined to be solution of the following *time-discretized* equation (the space discretization is not relevant in this framework), associated to some intermediate boundary conditions:

$$\begin{aligned}\left(I-\frac{\Delta t}{2Re}\nabla^2\right)\mathbf{v}^* &= \left(I+\frac{\Delta t}{2Re}\nabla^2\right)\mathbf{v}^n-\frac{\Delta t}{2}\underline{\nabla}\cdot(3\mathbf{v}^n\mathbf{v}^n-\mathbf{v}^{n-1}\mathbf{v}^{n-1}),\quad \mathbf{x}\in\Omega \\ \mathbf{v}^*(\mathbf{x}) &= \mathbf{v}^*_\partial,\quad \mathbf{x}\in\partial\Omega\end{aligned} \tag{21}$$

Then, the vector field $\mathbf{v}^*$, is decomposed according to

$$\mathbf{v}^* = \mathbf{v}^{n+1} + \underline{\nabla}\Phi^{n+1} \tag{22}$$

by solving the Poisson problem[††]

$$\begin{aligned} \nabla^2\Phi^{n+1} &= \underline{\nabla}\cdot\mathbf{v}^*, \quad \mathbf{x}\in\Omega \\ \mathbf{n}\cdot\underline{\nabla}\Phi^{n+1} &= \mathbf{n}\cdot(\mathbf{v}^*-\mathbf{v}^{n+1}), \quad \mathbf{x}\in\partial\Omega \end{aligned} \tag{23}$$

thus obtaining the vector field $\mathbf{v}^{n+1}=P_X(\mathbf{v}^*)$ (or, $\mathbf{v}^{n+1}=P_H(\mathbf{v}^*)$ when possible). This final divergence-free vector is expected to satisfy, at least up to second-order terms in time, the original coupled momentum and continuity equations system, as well as the boundary conditions.

As far as the relation between $\underline{\nabla}\Phi^{n+1}$ and $\underline{\nabla}p'$ is concerned, it is determined by combining $(21)_1$ as $(I-\frac{\Delta t}{2Re}\nabla^2)\mathbf{v}^{n+1}=(I-\frac{\Delta t}{2Re}\nabla^2)\mathbf{v}^*-\int_{t^n}^{t^{n+1}}\underline{\nabla}p'\,\mathrm{d}t$ and (22) so that it results

$$\Delta t\underline{\nabla}\langle p'\rangle^{n+1} \equiv \int_{t^n}^{t^{n+1}} \underline{\nabla}p'\,\mathrm{d}t = \left(I-\frac{\Delta t}{2\,Re}\nabla^2\right)\underline{\nabla}\Phi^{n+1} \tag{24}$$

Equation (24) highlights the fact that the gradient field $\underline{\nabla}\Phi^{n+1}$ always represents only a first-order approximation of the authentic *pressure gradient* which appears into the coupled momentum equation (9). It is interesting to exploit Equation (9) for expressing the (24) in terms of the orthogonal projector (as long as it is definable) as $\underline{\nabla}p'=(I-P_H)(\mathbf{a}^*)$ and $\underline{\nabla}\Phi^{n+1}=(I-P_H)(\mathbf{v}^*)$:

$$\begin{aligned} (I-P_H)\int_{t^n}^{t^{n+1}}\mathbf{a}^*\,\mathrm{d}t &= \left(I-\frac{\Delta t}{2\,Re}\nabla^2\right)[(I-P_H)(\mathbf{v}^*)] \\ &= (I-P_H)\left[\left(I-\frac{\Delta t}{2\,Re}\nabla^2\right)\mathbf{v}^*\right]+\frac{\Delta t}{2\,Re}[\nabla^2P_H(\mathbf{v}^*)-P_H(\nabla^2\mathbf{v}^*)] \end{aligned} \tag{25}$$

In general, if the domain is confined, commutation between Laplacian and gradients operators does not apply on the boundary, $P_H(\nabla^2\mathbf{v}^*)\neq\nabla^2P_H(\mathbf{v}^*)=\nabla^2\mathbf{v}^{n+1}$ since the vector $P_H(\nabla^2\mathbf{v}^*)$ is divergence-free and has a null normal component to the boundary whereas the vector $\nabla^2\mathbf{v}^{n+1}$ is divergence-free but not necessarily parallel to the boundary. By formally rewriting (25) as

$$\int_{t^n}^{t^{n+1}}\mathbf{a}^*\,\mathrm{d}t\equiv\Delta t\langle\mathbf{a}^*\rangle^{n+1} = \left(I-\frac{\Delta t}{2\,Re}\nabla^2\right)\mathbf{v}^*+\frac{\Delta t}{2\,Re}(I-P_H)^{-1}[\nabla^2\mathbf{v}^{n+1}-P_H(\nabla^2\mathbf{v}^*)] \tag{26}$$

one highlights the interesting aspect that the pressure gradient does not appear and whatever expression one adopts for the intermediate boundary condition $(21)_2$, it should satisfy Equation (26) on $\partial\Omega$.

[††]Observe that only if were possible to express $\mathbf{a}^*=\frac{\partial\mathbf{v}^*}{\partial t}$ then Equation (23) would result the counterpart of problem (17) when this latter is integrated in time, being $\underline{\nabla}\Phi^{n+1}=\int_{t^n}^{t^{n+1}}\underline{\nabla}p'\,\mathrm{d}t$.

Consider that in the PFP method $\mathbf{v}^*$ and $\Phi^{n+1}$ are only auxiliary variables, therefore the original coupled Navier–Stokes system does not specify how to prescribe their boundary values. However, as Remark 1 extends to problem (23), whatever boundary values for $\mathbf{v}^*$ has been prescribed in solving the prediction step (21), it is sufficient in solving (23) that its normal component equals the difference between the normal components of the gradient field and the exact velocity. In other words, in $(23)_2$ it suffices to know the correct velocity $\mathbf{n}\cdot\mathbf{v}^{n+1}$ whereas no one of the others needs to be singularly prescribed. Therefore, while ensuring continuity, the $(23)_2$ set the solution to exactly satisfy the correct normal flow condition.

On the other side, according to the potential character of the auxiliary variable $\Phi^{n+1}$, nothing can be ensured by solving (23) as far as the tangential component is concerned. The uncertainty arises from the rotational part of the solution, namely from the intermediate velocity obtained from (21). In fact, by projecting (22) along the tangential direction to the boundary, one gets $\mathbf{t}\cdot\mathbf{v}_\partial^{n+1} = \mathbf{t}\cdot(\mathbf{v}_\partial^* - \underline{\nabla}\Phi|_\partial^{n+1})$ wherein $\mathbf{t}\cdot\mathbf{v}_\partial^*$ is the boundary value already prescribed in solving problem (21). Actually, the PFP procedure requires that, before $\Phi^{n+1}$ is available, some functional relation in $(21)_2$, generally of the type $\mathbf{v}_\partial^* = \mathbf{v}_\partial^{n+1} + \mathbf{f}(\Phi)|_\partial$, has been already prescribed. As a consequence, in order for the tangential component to be satisfied (at least at the same accuracy resulting in the interior) one should ensure that $\mathbf{t}\cdot(\mathbf{f}(\Phi)|_\partial - \underline{\nabla}\Phi|_\partial^{n+1}) = O(\Delta t^3)$ verifies at the end of the steps. Let us remark that the homogeneous condition $\mathbf{f}(\Phi)|_\partial = \mathbf{0}$ has been often used in many papers.

From the previous considerations, it should be clear that $\mathbf{v}^*$ has to be considered only as a mathematical position. As a matter of fact, by hypothesizing that a continuous-in-time field $\mathbf{v}^*(t)$ exists, Kim and Moin [10] following the procedure of Oliger and LeVeque, see Reference [35], proposed to specify, by means of a Taylor series expansion about $t^n$, a non-homogeneous intermediate boundary conditions $(21)_2$, i.e. $\mathbf{f}(\Phi)|_\partial = \underline{\nabla}\Phi|_\partial^n$. Observe that, by doing so it was implicitly assumed to distinguish two time levels for the provisional field, i.e. $\mathbf{v}^{*n+1}$ and $\mathbf{v}^{*n}$. Such type of boundary has been widely used as it is supposed to produce results that are more accurate.

However, when the intermediate boundary conditions are those proposed by Kim and Moin, one comes into two possible kinds of errors:

1. A numerical boundary layer owing to the fact that $\partial\Phi/\partial n|_\partial = \text{const}(t)$ is implicitly consequential.
2. A slip condition error as it results $\mathbf{t}\cdot(\underline{\nabla}\Phi|_\partial^n - \underline{\nabla}\Phi|_\partial^{n+1}) = O(\Delta t)$.

Several analyses of such errors were performed and reported in References [19–28]. For example, E and Liu [20] reported in 1995: *it has been a mystery for twenty-five years that the projection method seems to perform better than expected.* They stated that the effect of solid boundaries does not result restricted to create numerical boundary layers but they introduce high-frequency oscillations reducing the order of the accuracy in the interior of the domain. Specifically, Dirichlet boundary conditions for pressure were shown to lead to $O(1)$ numerical boundary layers. They performed the analysis by prescribing $\partial p/\partial n|_\partial^{n+1} = 0$ while considering $\mathbf{n}\cdot\mathbf{v}_\partial^* = 0$ in the prediction step. However, in this way, the differences from the real pressure gradient and the auxiliary gradient field defined into (22) are not highlighted but it was stated that the gradient field solution of (23) directly approximates the authentic pressure gradient, which therefore has a numerical boundary layer. More recently, Strikwerda and Lee [23] stated from their analysis that the numerical boundary layer really is in the auxiliary variable $\Phi$, not in the pressure. Ultimately, Brown *et al.* [14] showed that a first-order convergence in the pressure is numerically obtained, despite of the results provided by the normal mode analysis that predicts, whatever prescribed value $\mathbf{n}\cdot\mathbf{v}_\partial^*$ as long as respecting (23), second-order convergence. They adduced this lack in the convergence rate to the fact that the field $\mathbf{v}^*$ is not smooth close to the boundary so resulting also for the variable $\Phi$ and in order for this error to be remedied, they proposed to adopt inhomogeneous Neumann boundary conditions. However, owing to the peculiar construction of the PFP method, it can be shown that the knowledge of the authentic pressure gradient field is never required to retain the second-order accuracy in the velocity, at least in case of prescribed periodical condition. It is generally accepted that, if one adopts the *pressure-free projection method*, the limited accuracy of the gradient does not limit the accuracy in the velocity vector $\mathbf{v}^{n+1}$ because at any time step the pressure error is disregarded [14] by prohibiting that it could accumulate in time and contribute in the momentum equation solution. For different prescription of the boundary conditions (e.g. confined flows), in order to retain fully second-order accuracy all the way up to the boundary, one must proceed in such a way that the produced numerical boundary layer does not affect the velocity (or equivalently to require orthogonality of the decomposition) while obtaining sufficiently smooth slip condition. Hence, it is suggested that if the boundary layer mode is an exact gradient it does not contribute to the divergence-free velocity field. Again, as reported by E and Liu [20], '*the question to be addressed is whether orthogonality is really important*' in this context. Thus, a crucial issue consists in analysing if orthogonality and uniqueness of the decomposition are really necessary for remedying to the errors.

Moreover, to the best of the author's knowledge, a specific discontinuity feature of the auxiliary variable $\underline{\nabla}\Phi$ does not seem to have been sufficiently highlighted in the literature. In fact, the superscript notation $n$ in $\mathbf{f}(\Phi)|_\partial = \underline{\nabla}\Phi|_\partial^n$ is someway misleading as the gradient of the field $\Phi$, as well as the intermediate field $\mathbf{v}^*$, is not a continuous function in time. As a matter of fact, in the considered time interval $(t^n, t^{n+1})$ in which one wants to determine $\mathbf{v}^*$, because the intermediate field was set as $\mathbf{v}^{*n} = \mathbf{v}^n$ (namely its previous value is disregarded and it is reset to the initial divergence-free velocity) according to decomposition (22) it should congruently result $\underline{\nabla}\Phi^n = \mathbf{0}$. Therefore, in the integration interval $(t^n,\, t^{n+1})$, one must be aware that notation $\underline{\nabla}\Phi^n$ does not stand for the value of the gradient field at the time $t^n$ but it is that one computed in the previous time-integration step in the interval $(t^{n-1}, t^n)$ thus, to be more properly indicated as $\underline{\nabla}\Phi^{n^-}$. As a conclusion, one can deduce that different limiting values $\underline{\nabla}\Phi^{n^-} \neq \underline{\nabla}\Phi^{n^+} = \mathbf{0}$ exist. A major extension of this boundary condition is plausible provided that this issue is considered.

Now, defining the splitting error vector fields according to the expressions $\mathbf{e}_v^{n+1} = \tilde{\mathbf{v}}^{n+1} - \mathbf{v}^{n+1} = \mathrm{e}^{\Delta t \frac{\partial}{\partial t}|^n}\tilde{\mathbf{v}}(t) - (\mathbf{v}^* - \underline{\nabla}\Phi^{n+1})$ and $\mathbf{e}_p^{n+1} = (\underline{\nabla}\tilde{p}'^{n+1} - \underline{\nabla}\Phi^{n+1})$ where $\tilde{\mathbf{v}}^{n+1}$, $\underline{\nabla}\tilde{p}'^{n+1}$ are the exact fields and $\mathrm{e}^{\Delta t\partial/\partial t|^n}$ is the solution operator and projecting $\mathbf{e}_p^{n+1}$ along the velocity it is easy to verify that if and only if $\mathbf{e}_p^{n+1} \sim \underline{\nabla}\chi$ then $\int_\Omega \mathbf{e}_p^{n+1}\cdot\mathbf{v}^{n+1}\,\mathrm{d}V = \int_{\partial\Omega}\mathbf{n}\cdot\mathbf{v}^{n+1}\chi\,\mathrm{d}S$ and such integral vanishes within the same constraint of orthogonality. In the PFP, according to (24) if the domain is confined, commutation between Laplacian and gradients operators in general does not apply on the boundary allowing the pressure error to enter into the velocity accuracy. If this is the case, one should ensure that the error has not a magnitude order greater than in the interior.

The next section is devoted to make clearer such issues by performing a numerical validation and highlighting the consequences of the lack into orthogonality of the decomposition.

## 5. NUMERICAL VALIDATION

For the sake of clarity, these aspects are highlighted in the following example of the 2-D Taylor solution of the Navier–Stokes system:

$$
\begin{aligned}
u(x,y,t) &= -\cos x \sin y\, \mathrm{e}^{-2t} \\
v(x,y,t) &= \sin x \cos y\, \mathrm{e}^{-2t} \\
p(x,y,t) &= -0.25(\cos 2x + \cos 2y)\, \mathrm{e}^{-4t}
\end{aligned} \tag{27}
$$

Despite the fact that this simple solution embodies a separate equilibrium between acceleration and diffusion, convection and pressure gradient, the attractive feature of such a solution consists in the possibility to prescribe Dirichlet or Neumann boundary conditions along fictitious finite domains, which can be chosen in such a way to test orthogonality. According to such idea, two domains will be fixed, the one verifying orthogonality, the other not.

Let us first assume the bounded domain $\Omega_1 = [-\pi\times\pi]\times[-\pi\times\pi]$ for which $\mathbf{n}\cdot\mathbf{a} = a_n \neq 0$ results from $(27)_{1,2}$ evaluated on the frontier $\partial\Omega_1$. Nevertheless, it is easy to verify that $\int_{\Omega_1} \mathbf{a}\cdot\underline{\nabla} p'\,\mathrm{d}V = 0$ and therefore Equation (9) will express an orthogonal decomposition of $\mathbf{a}^*$. However, such decomposition is not unique, existing two distinct solutions of problems (16) and (17) even if the source terms are the same. In fact, from Equation $(27)_3$ it results that the Neumann boundary conditions for (17) turns to be homogeneous differently from the boundary condition for (16). Observe that the decomposition (9) would return to be both orthogonal and unique if the bounded domain were extended, for example, to $[-\pi/2\times\pi/2]\times[-\pi/2\times\pi/2]$ since for it $\mathbf{n}\cdot\mathbf{a} = 0$ on the frontier. The second chosen bounded domain is $\Omega_2 = [0\times\pi]\times[0\times 1]$, for which results $\int_{\Omega_2} \mathbf{a}\cdot\underline{\nabla} p'\,\mathrm{d}V \neq 0$. The situation is sketched into the vector plots, reported in Figures 1, obtained at time $t = 0$ from Equation (27), for (1a) the Eulerian acceleration $\mathbf{a}$, (1b) the pressure gradient field $\underline{\nabla} p'$, respectively. Owing to the tangential direction of the pressure gradient along the frontier $\partial\Omega_1$, perpendicularity between the vectors field, i.e. $\mathbf{a}\cdot\underline{\nabla} p' = 0$, also subsists. On the other hand, along the frontier $\partial\Omega_2$, perpendicularity between the vectors field does no longer subsist at boundary $y = 1$.

The numerical results reported in this section concerns the application of the HHD for determining:

(a) The Eulerian acceleration (9) at $t_0$ for integrating in time according to (11) and obtaining the divergence-free velocity field and pressure gradient at $T$.
(b) The divergence-free velocity field (22) and pressure gradient at $T$ after that an intermediate non-solenoidal velocity, based on the PFP procedure (21), is computed.

A second-order co-located finite difference (FD) approximation was adopted and the splitting errors $(\tilde{\mathbf{v}}^{n+1} - \mathbf{v}^{n+1})$ and $(\underline{\nabla}\tilde{p}'^{n+1} - \underline{\nabla} p'^{n+1})$ are computed in the $L_\infty$ norm over the domains, including the boundaries, after a single time step. Applying a standard SOR procedure, the algebraic linear systems are solved. The number of grid points is $(315\times 315)$ for $\Omega_1$ and $(158\times 51)$ for $\Omega_2$, respectively thus, the same uniform mesh size $h = O(10^{-2})$ acts for all tests. In one time step, the splitting error (or equivalently, *discretization error*) for the velocity i.e. the difference between the exact $(27)_{1,2}$ and the numerical solution, relates to the LTE by the time step (for more details see Reference [36]) and this concept is now exploited to analyse the splitting. Hence, the splitting errors accords with $e_s = \Delta t\, O(\mathrm{LTE})$ standing LTE for the *Local Truncation Error* which is $O(\Delta t^p, h^2)$, whose magnitude order $p$ will define the actual

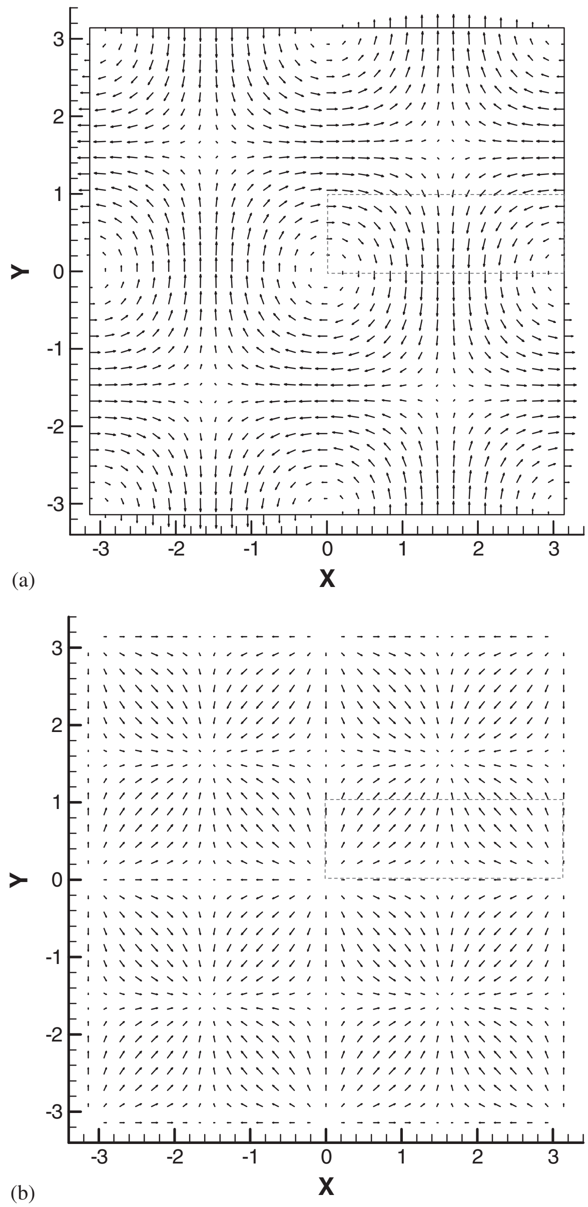



Figure 1. (a) Vector plot of the Eulerian acceleration **a** expressed from Equation $(27)_{1,2}$ at $t=0$ and represented in the finite domain $\Omega_1=[-\pi,\pi]\times[-\pi,\pi]$ wherein the decomposition is orthogonal. The dotted box $[0,\pi]\times[0,1]$ represents the finite domain $\Omega_2$ in which the decomposition is not orthogonal. (b) Vector plot of the pressure gradient field expressed from Equation $(27)_3$ at $t=0$ and represented in $\Omega_1=[-\pi,\pi]\times[-\pi,\pi]$. For a better visualisation, the vector length has also been increased twice than that in (a). Note that the boundaries plot has been removed from the figure for highlighting the tangential direction of the pressure gradient along the frontier $\partial\Omega_1$ whereas perpendicularity between the vectors field, i.e. $\mathbf{a}\cdot\underline{\nabla}p'=0$, subsists. On the other hand, along the frontier $\partial\Omega_2$ perpendicularity between the vectors field does not subsist at $y=1$.

accuracy of the projection method. Thus, the accuracy order of the method is defined by the rate at which the LTE goes to zero for vanishing integration parameters. Thus, after that the splitting errors is computed, by simply dividing it for the time step, the value $p$ associated to the projection method can be deduced in a straightforward way. Let us begin to illustrate the results obtained for the domain $\Omega_1$. The plots of the error curves are shown in Figures 2 and 3 for the cases addressed in (a) and (b), respectively.

For what regards with case (a), the convergence rates for the $u$ and $v$ velocity components are reported in Figure 2(a) while those for $\partial p/\partial x$ and $\partial p/\partial y$ components are reported in Figure 2(b). The initial third-order slope of the velocities convergence confirms the second-order accuracy (i.e. $p=2$) of the solution all the way up to the boundary. Let us remind that the computations are performed with a variable Courant number $|\mathbf{v}|\Delta t/h$ and the reported errors for the velocity, Figure 2(a), relates to the LTE according to $(\Delta t^{p+1}, \Delta t\, h^2)$. Therefore, it appears that, for vanishing time steps and fixed $h$, the slope changes according to a first-order asymptotic convergence when the time step becomes comparable to the mesh size. The same features appear in Figure 2(b) but, owing to the gradient space-discretization, the rate of convergence tends to the constant second-order $h^2$ value.

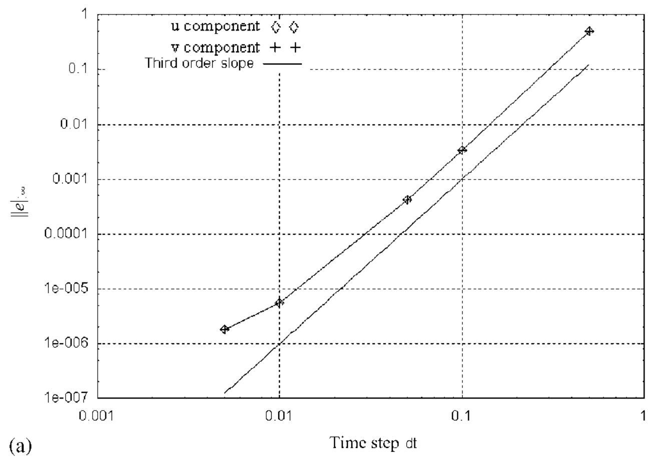
For what regards with case (b), the convergence rates for the $u$ and $v$ velocity components are reported in Figure 3(a) while those for $\partial\Phi/\partial x$ and $\partial\Phi/\partial y$ components are reported in 3(b). However, now the resulting initial second-order slope of the velocity convergence indicates the first-order accuracy (i.e. $p=1$) of the solution all the way up to the boundary. This decreasing into the accuracy is caused by the error in the tangential component introduced by the intermediate auxiliary boundary condition $\mathbf{f}(\Phi)|_\partial = \underline{\nabla}\Phi|_\partial^n$ that cannot be corrected by the projection step (23). In fact, it was verified that the maximum error remains always localized on the boundaries. It is worthwhile remarking that by performing the same computation but for bi-periodical boundary conditions, the third-order slope was restored. As before, the reported errors for the velocities, Figure 3(a), relates to the LTE therefore, for vanishing time steps, the slope must change. According to (24), the convergence for $\partial\Phi/\partial x$ and $\partial\Phi/\partial y$ components is also reduced. Owing to the gradient discretization, the rate of convergence tends to the constant second-order $h^2$ value.

Let us now illustrate the results obtained for the non-orthogonal decomposition into the finite domain $\Omega_2$. The plots of the error curves are shown in Figures 4 and 5 for the cases addressed in (a) and (b), respectively.

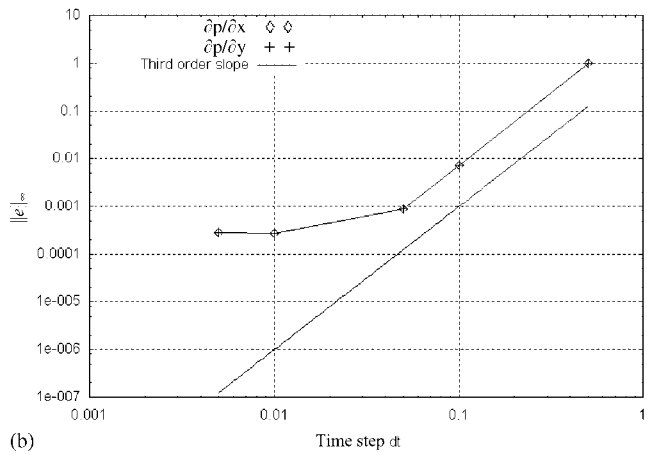
For what regards with case (a), the convergence rates for the $u$ and $v$ velocity components are reported in Figure 4(a) while those for $\partial p/\partial x$ and $\partial p/\partial y$ components are reported in 4(b). Now, although the third-order slope of the velocities convergence in Figure 4(a) confirms the second-order accuracy (i.e. $p=2$) of the solution, the pressure gradient convergence rates in Figure 4(b) shows a drastic reduction. This specific effect is fundamental in understanding the modal interaction between velocity and pressure. In a single time step the illustrated effects remains clearly separated but when one reiterates the time integration starting from the obtained solution at $T$, then the pressure errors tend to enter into the velocity accuracy.

For what regards with case (b), the convergence rates for the $u$ and $v$ velocity components are reported in Figure 5(a) while those for $\partial\Phi/\partial x$ and $\partial\Phi/\partial y$ components are reported in 5(b). However, now in Figure 5(a) the initial second-order slope tends to become first order for $\Delta t < 0.1$ but this time, the error level for which this happens is not the $O(h^2)$ of the LTE. What is more, consists into the evident lack of convergence in Figure 5(b) concerning the auxiliary pressure gradient. It is worthwhile remarking that to these differences in results




(a)




(b)

Figure 2. (a) Determination of the Eulerian acceleration $\mathbf{a}_0$ by means of the orthogonal decomposition (9) in the domain $\Omega_1 = [-\pi, \pi] \times [-\pi, \pi]$ and second-order time integration (11) for determining $\mathbf{v}^{n+1}$. Convergence rate for the $u$ and $v$ velocity components. The errors, computed in the $L_\infty$ norm are shown against the time step in a double logarithmic scale. The third-order slope confirms the second-order accuracy of the solution all the way up to the boundary. (b) Determination of the Eulerian acceleration $\mathbf{a}_0$ by means of the orthogonal decomposition (9) in the domain $\Omega_1 = [-\pi, \pi] \times [-\pi, \pi]$ and second-order time integration (11) for determining $\mathbf{v}^{n+1}$. Convergence rate for the $\partial p/\partial x$ and $\partial p/\partial y$ components. The errors, computed in the $L_\infty$ norm are shown against the time step in a double logarithmic scale. Owing to the gradient discretization the rate of convergence tends to the constant second-order $h^2$ value.

reported in Figures 4(a) and 5(a), contribute the fact the in the latter case the time integration (21) is implicit, therefore also in one time step the computation of $\mathbf{v}^*$ already suffers by the pressure errors whereas the explicit time integration in the former retards such effect which would be evident during the successive time integration.

It is consequent to ascribe the main responsibility of the accuracy decreasing in the lack of the orthogonality of decomposition (9) and (22). The fundamental conclusion is that

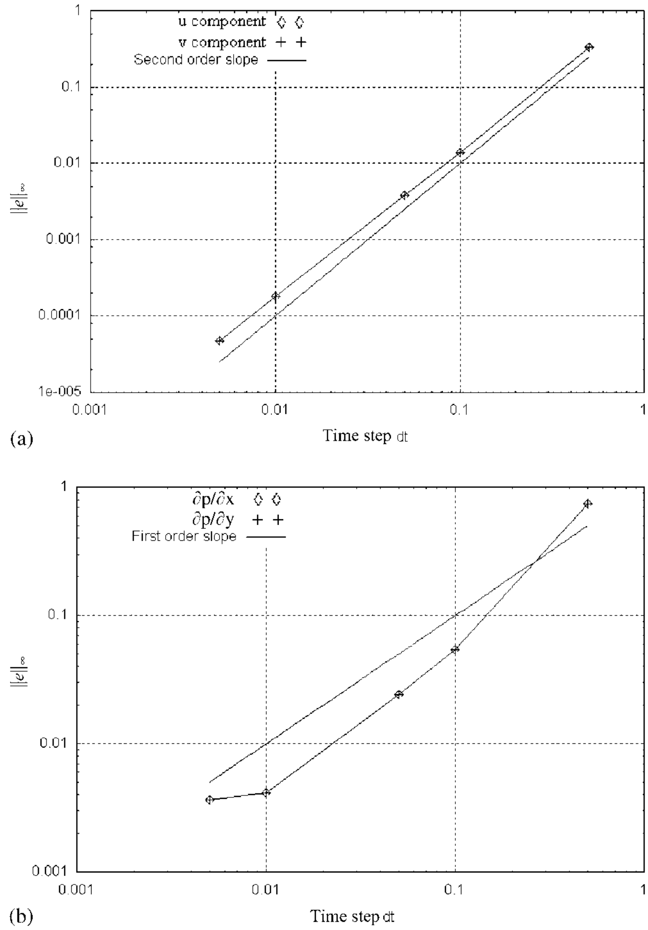



Figure 3. (a) Determination of the velocity $\mathbf{v}^{n+1}$ by means of the orthogonal decomposition (22) in the domain $\Omega_1 = [-\pi, \pi] \times [-\pi, \pi]$ and after second-order time integration (21) with the Kim-Moin boundary conditions for determining $\mathbf{v}^*$. Convergence rate for the $u$ and $v$ velocity components. The errors, computed in the $L_\infty$ norm are shown against the time step in a double logarithmic scale. The second order slope assess the first order accuracy of the solution all the way up to the boundary. This decreasing into the accuracy is caused by the error in the tangential component introduced by the intermediate boundary condition. (b) Determination of the velocity $\mathbf{v}^{n+1}$ by means of the orthogonal decomposition (22) in the domain $\Omega_1 = [-\pi, \pi] \times [-\pi, \pi]$ and after second-order time integration (21) with the Kim-Moin boundary conditions for determining $\mathbf{v}^*$. Convergence rate for the $\partial\Phi/\partial x$ and $\partial\Phi/\partial y$ components. The errors, computed in the $L_\infty$ norm are shown against the time step in a double logarithmic scale. Owing to the gradient discretization the rate of convergence tends to the constant second-order $h^2$ value.

orthogonality of the decomposition should be always maintained for all the flow problems of practical interest thus, Case II should be considered as guideline for performing an accurate projection.

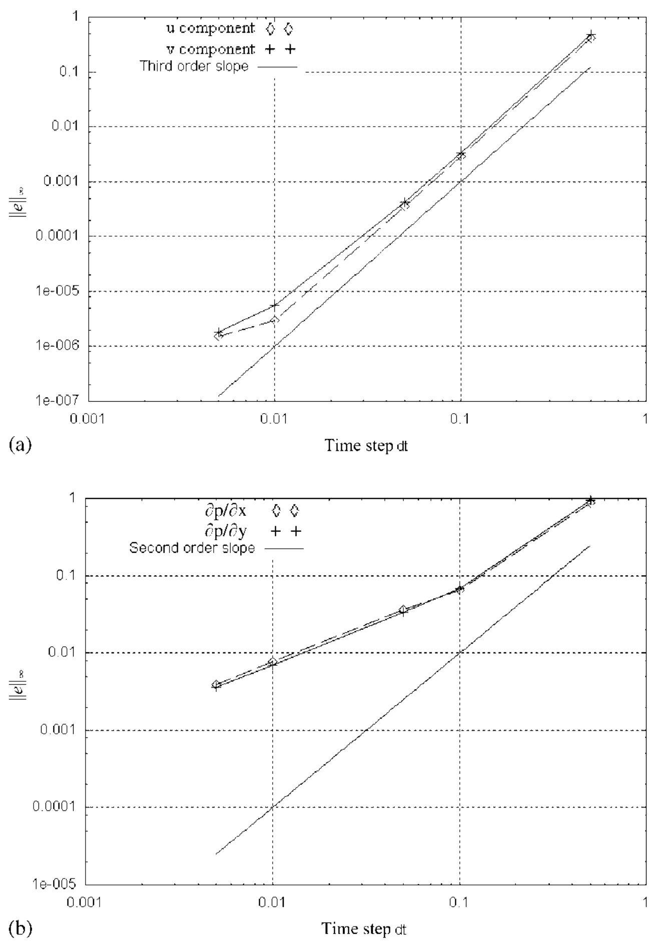



Figure 4. (a) Determination of the Eulerian acceleration $\mathbf{a}_0$ by means of the non-orthogonal decomposition (9) in the domain $\Omega_2 = [0, \pi] \times [0, 1]$ and second-order time integration (11) for determining $\mathbf{v}^{n+1}$. Convergence rate for the $u$ and $v$ velocity components. The errors, computed in the $L_\infty$ norm are shown against the time step in a double logarithmic scale. The third-order slope confirms the second-order accuracy of the solution all the way up to the boundary. (b) Determination of the Eulerian acceleration $\mathbf{a}_0$ by means of the non-orthogonal decomposition (9) in the domain $\Omega_2 = [0, \pi] \times [0, 1]$ and second order time integration (11) for determining $\mathbf{v}^{n+1}$. Convergence rate for the $\partial p/\partial x$ and $\partial p/\partial y$ components. The errors, computed in the $L_\infty$ norm are shown against the time step in a double logarithmic scale. Comparing with Figure 2(b), it clearly appears the reduced slope.

## 6. CONCLUSIONS

The role of the Helmholtz–Hodge decomposition theorem in being a part of projection methods adopted for solving the Navier–Stokes system for incompressible flows with prescribed general boundary conditions is illustrated. As, in a bounded domain, it is required that the divergence-

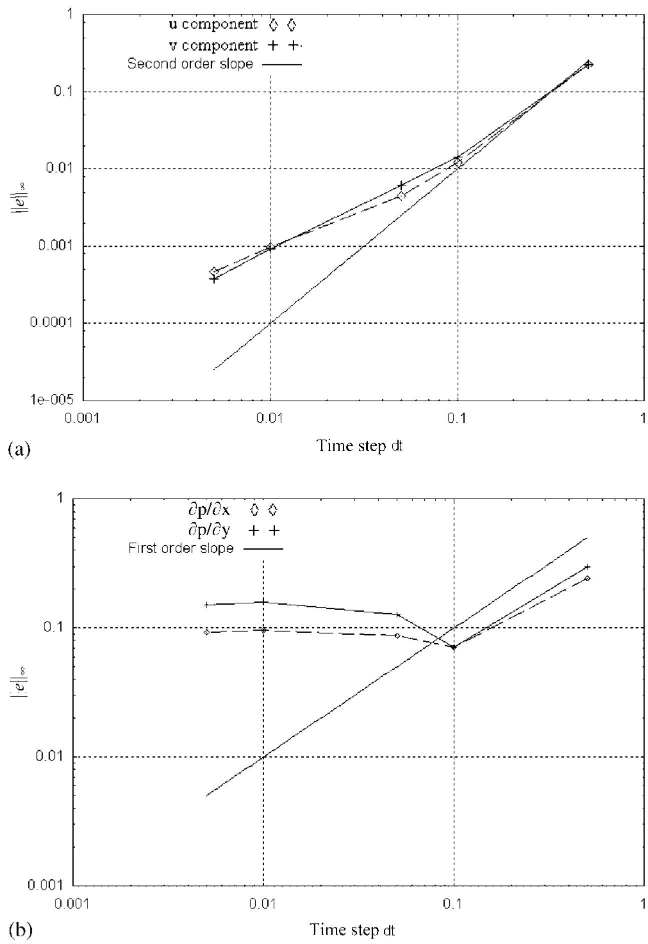



Figure 5. (a) Determination of the velocity $\mathbf{v}^{n+1}$ by means of the non-orthogonal decomposition (22) in the domain $\Omega_2 = [0, \pi] \times [0, 1]$ and after second-order time integration (21) with the Kim-Moin boundary conditions for determining $\mathbf{v}^*$. Convergence rate for the $u$ and $v$ velocity components. The errors, computed in the $L_\infty$ norm are shown against the time step in a double logarithmic scale. The initial second order slope tends to become first order for $\Delta t < 0.1$. Comparing with Figure 3(a), it clearly appears the reduced slope. (b) Determination of the velocity $\mathbf{v}^{n+1}$ by means of the non-orthogonal decomposition (22) in the domain $\Omega_2 = [0, \pi] \times [0, 1]$ and after second-order time integration (21) with the Kim-Moin boundary conditions for determining $\mathbf{v}^*$. Convergence rate for the $\partial\Phi/\partial x$ and $\partial\Phi/\partial y$ components. The errors, computed in the $L_\infty$ norm are shown against the time step in a double logarithmic scale. The initial first-order slope tends to become constant for $\Delta t < 0.1$.

free vector field belongs to a subspace of vector fields parallel to the frontier, it was shown that the decomposition can be applied in a more general way in the determination of the Eulerian acceleration rather than of the velocity. On the other hand, the consequences of adopting projection methods with general boundary conditions were also addressed in terms of the lack into the orthogonality and uniqueness of the decomposition. An approach for determining the Eulerian acceleration vector field with a non-vanishing normal component via two-successive orthogonal decompositions is addressed. An answer to the question if orthogonality is really important is consequently provided. The conclusions are then extended to the *pressure-free projection* method.

Moreover, in principle, since the Helmholtz–Hodge decomposition in a bounded domain requires only one condition to be well posed, it is not possible to satisfy all the boundary conditions for the projected vector field. It was clarified that, if on a side the original Stokes-like system allows to correctly impose all the boundary conditions, the pressure-free (de-coupled) system is based on two steps: in the first an auxiliary non-solenoidal vector field is determined from an implicit parabolic-type equation associated to normal and tangential components for it to be prescribed before resolving the projection step. In the second step, the projection requires only one condition in terms of the physical normal component of the velocity that corrects that previously assigned in the first step. Therefore, globally, the de-coupled system requires the same number of boundary conditions of the original system but apart from that assigned during the projection, the exactness of the others is not ensured.

As a conclusion, this study is part of a more general analysis concerning the development of high-order accuracy projection-based methods for developing Large Eddy Simulation for turbulence [37]. It is therefore necessary to re-interpret the boundary conditions $(21)_2$ when the intermediate vector velocity is a filtered quantity and $(21)_1$ contains an SGS model. This fact forces the form of boundary conditions for the tangential components to be derived from an equivalent filtered counterpart. Therefore, future work will be addressed in the future.

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